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# ON THE MULTIPLICATIVE SEMIGROUP OF A RING

BY

A. A. KLEIN

*School of Mathematical Sciences, Sackler Faculty of Exact Sciences*
*Tel Aviv University, Tel Aviv, Israel 69978*
*e-mail: aaklein@math.tau.ac.il*

AND

H. E. BELL*

*Department of Mathematics, Brock University*
*St. Catharines, Ontario, Canada L2S3A1*
*e-mail: hbell@spartan.ac.brocku.ca*

ABSTRACT

We prove that a PI-ring is finite if its multiplicative semigroup is finitely-generated.

Euclid's Theorem on the existence of infinitely many primes may be stated as follows: The multiplicative semigroup of the ring of integers is not finitely-generated. In a very pretty paper [3], Isbell has proved that the same is true of the multiplicative semigroup of any infinite *commutative* ring; however, whether it is true for arbitrary infinite rings is an open question. Our purpose is to provide an affirmative answer for PI-rings. We prove

THEOREM: *If $R$ is a PI-ring in which $(R, \cdot)$ is finitely-generated, then $R$ is finite.*

We follow a familiar strategy of proof. We begin with the case of prime rings, then proceed to semiprime rings and finally to arbitrary PI-rings.

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LEMMA 1: *Let $D$ be a noncommutative division ring which is finite-dimensional over its center $F$. Then $F$ contains a free multiplicative abelian group of infinite rank.*

*Proof:* The result is clear if $\operatorname{char} F = 0$, for then $F$ contains the multiplicative group of positive rationals. If $\operatorname{char} F = p > 0$, then by a theorem of Jacobson [4, I, p. 315], $F$ must contain an element $t$ which is transcendental over the prime subfield $\mathbb{Z}_p$; hence $F$ contains the field $\mathbb{Z}_p(t)$, which contains a free multiplicative abelian group of infinite rank.

LEMMA 2: *Let $R$ be a prime PI-ring. If $(R, \cdot)$ is finitely-generated, then $R$ is a total matrix ring over a finite field.*

*Proof:* Since $R$ is a prime PI-ring, it has nontrivial center $Z$ and its central localization $R_Z$ is a primitive PI-ring [4, Th. 6.1.30]. Moreover, the elements of $Z^* = Z\backslash\{0\}$ are invertible in $R_Z$. It follows that $(R_Z, \cdot)$ is finitely-generated, for if $b_1, b_2, \ldots, b_r$ generate $(R, \cdot)$ and $b_1, b_2, \ldots, b_s$ are the $b_i$'s which are invertible in $R_Z$, then the elements of $Z^*$ may be expressed as products of $b_1, \ldots, b_s$, so that $b_1, \ldots, b_r, b_1^{-1}, \ldots, b_s^{-1}$ generate $(R_Z, \cdot)$. If $R_Z$ is finite, then $R$ is finite and hence $R = R_Z$; therefore, we may assume that $R$ is primitive.

By Kaplansky's Theorem [4, Th. 6.1.25], $R \cong M_k(D)$, where $D$ is a division ring finite-dimensional over its center $F$. Let $\dim_F R = n$, and view $R$ as a subring of $M_n(F)$. Assume $(R, \cdot)$ is generated by $a_1, a_2, \ldots, a_r$, with $a_1, a_2, \ldots, a_s$ invertible and $a_{s+1}, \ldots, a_r$ noninvertible. Consider the multiplicative semigroup homomorphism $\phi: R \to F$ given by $\phi(a) = \det a$. Since $\det a_i = 0$ for $i = s+1, \ldots, r$, $\operatorname{im}\phi$ is generated by $\det a_1, \ldots, \det a_s$ and 0. For an element $a \in F^* = F\backslash\{0\}$, $\det a = a^n \neq 0$, so $\langle \det a : a \in F^* \rangle$ is contained in the abelian group generated by $\det a_1, \ldots, \det a_s$ and therefore $F^*$ does not contain a free abelian group of infinite rank. By Lemma 1 and its proof, we see that $D = F$, $\operatorname{char} F = p > 0$, and $F$ is algebraic over $\mathbb{Z}_p$. It follows that $\det a_1, \ldots, \det a_s$ generate a finite field, so that the elements $\det a, a \in F^*$, have bounded order and so do the elements of $F^*$. Therefore, $F$ is finite.

The extension from prime to semiprime rings is surprisingly involved, and it is preceded by a lemma.

Note that if $R$ is a subdirect product of $(R_\alpha)_{\alpha \in A}$ and $B$ is a nonvoid subset of $A$, then $R$ has a homomorphic image which is a subdirect product of $(R_\beta)_{\beta \in B}$, namely $R / \bigcap_{\beta \in B} \ker \pi_\beta$. This ring may be obtained by applying the projection $\Pi_{\alpha \in A} R_\alpha \to \Pi_{\beta \in B} R_\beta$ on $R$.

In the next result we shall say that an element $a \in \Pi_{\alpha \in A} R_\alpha$ is regular (singular) on a subset $B$ of $A$ if $a(\beta)$ is regular (singular) in $R_\beta$ for each $\beta \in B$.

LEMMA 3: *Let $R$ be a ring in which $(R,\cdot)$ is finitely-generated; and suppose that $R$ is a subdirect product of $(R_\alpha)_{\alpha\in A}$, where for each $\alpha \in A$, $R_\alpha = M_{n_\alpha}(F_\alpha)$ for some finite field $F_\alpha$. Then $\{|F_\alpha| : \alpha \in A\}$ is bounded.*

*Proof:* Suppose on the contrary that $\{|F_\alpha| : \alpha \in A\}$ is not bounded, so there exists a sequence $\beta_1, \beta_2, \dots$ in $A$ with $|F_{\beta_i}| = m_i \to \infty$. Apply the projection $\Pi_{\alpha\in A} R_\alpha \to \Pi_{i=1}^\infty R_{\beta_i}$ on $R$ to obtain a ring $R'$ which is a subdirect product of $(R_{\beta_i})_{i=1}^\infty$, and for which $(R',\cdot)$ is finitely-generated, say by $a_1, a_2, \dots, a_r$. For $j = 1, 2, \dots, r$, let $B_j = \{\beta_i : a_j(\beta_i) \text{ is regular in } R_{\beta_i}\}$.

We proceed to show that there exists an infinite subset $C$ of $\{\beta_1, \beta_2, \dots\}$ such that some of the elements $a_1, \dots, a_r$ are regular on $C$ and the others are singular on $C$. Let $C_1 = B_1$ if $B_1$ is infinite and $C_1 = \{\beta_1, \beta_2, \dots\}\backslash B_1$ otherwise; let $C_2 = C_1 \cap B_2$ if $C_1 \cap B_2$ is infinite and $C_2 = C_1 \backslash B_2$ otherwise; and let $C_3 = C_2 \cap B_3$ if $C_2 \cap B_3$ is infinite and $C_3 = C_2 \backslash B_3$ otherwise. Continuing in this way, we finally get an infinite set $C = C_r$ such that for $j = 1, \dots, r$, $a_j$ is either regular on $C$ or singular on $C$.

Now applying the projection $\Pi_{i=1}^\infty R_{\beta_i} \longrightarrow \Pi_{\beta_i\in C} R_{\beta_i}$ on $R'$, we obtain a ring $R''$ such that $(R'',\cdot)$ is generated by the images of $a_1, a_2, \dots, a_r$, which we denote by $b_1, b_2, \dots, b_r$ respectively. Since elements of $R''$ are products of the $b_j$'s and each $b_j$ is either regular on $C$ or singular on $C$, it follows that each $b \in R''$ is either regular on $C$ or singular on $C$.

Choose $\beta_k \in C$ and let $|\,\mathrm{GL}(n_{\beta_k}, F_{\beta_k})| = t$. Since $|F_{\beta_i}| = m_i \to \infty$ and $C$ is infinite, there exists $\beta_\ell \in C$ such that $m_\ell > 2t$. Let $u$ be a generator of $F^*_{\beta_\ell}$; and note that since $R''$ is a subdirect product of $(R_{\beta_i})_{\beta_i\in C}$, $R''$ contains an element $b$ with $b(\beta_\ell) = u$. Since $b(\beta_\ell)$ is regular, $b$ is regular on $C$; in particular, $b(\beta_k)$ is regular and hence $b(\beta_k)^t = 1$. Now $b(\beta_\ell)^t = u^t \neq u^{2t} = b(\beta_\ell)^{2t}$, since the order of $u$ in $F^*_{\beta_\ell}$ is $m_\ell - 1 \geq 2t$; therefore, $b^t(\beta_\ell) - b^{2t}(\beta_\ell)$ is in $F^*_{\beta_\ell}$ and $b^t = b^{2t}$ is regular on $C$. But $b^t(\beta_k) - b^{2t}(\beta_k) = 1 - 1 = 0$, so we have a contradiction.

We are now ready to pass from prime rings to semiprime rings.

LEMMA 4: *Let $R$ be a semiprime PI-ring. If $(R,\cdot)$ is finitely-generated, then $R$ is finite.*

*Proof:* Let $d$ be the degree of a polynomial identity of $R$, and let $R$ be a subdirect product of a family of prime rings $(R_\alpha)_{\alpha\in A}$. For each $\alpha \in A$, $R_\alpha$ is a PI-ring with $(R_\alpha,\cdot)$ finitely-generated; therefore by Lemma 2, $R_\alpha \cong M_{n_\alpha}(F_\alpha)$, where $F_\alpha$ is a finite field and $n_\alpha \leq [d/2]$. By Lemma 3, $\{|F_\alpha \mid : \alpha \in A\}$ is finite, hence $\{F_\alpha : \alpha \in A\}$ is finite; and since $\{n_\alpha : \alpha \in A\}$ is clearly finite, $\{M_{n_\alpha}(F_\alpha) : \alpha \in A\}$ is finite. Finiteness of $R$ now follows from [2, Th. II.10.16].

*Proof of Theorem:* Let $N$ be the nil radical of $R$, and let $a_1, a_2, \ldots, a_r$ generate $(R, \cdot)$. By Lemma 4, the semiprime ring $R/N$ is finite; and we let $|R/N| = n$. Now since $R$ is a finitely-generated PI-ring, $N$ is nilpotent [1], say of index $k$; and since $na_1, \ldots, na_r \in N$, $n^k a_{i_1} \cdots a_{i_k} = 0$ for any $1 \leq i_1, \ldots, i_k \leq r$. This shows that $n^k R$ is finite.

Since $R/N$ is finite and $N^k = \{0\}$, there exist distinct positive integers $p, q$ such that $(x^p - x^q)^k = 0$ for all $x \in R$; thus $R$ is integral and so is $R/n^k R$. Applying Shirshov's Theorem [4, Th. 6.3.23], we see that $(R/n^k R, +)$ is finitely-generated; and being a torsion group, it must be finite. We conclude that $R$ is finite.

## References


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